

PURE O -SEQUENCES AND MATROID h -VECTORS

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ABSTRACT. We study Stanley’s long-standing conjecture that the h -vectors of matroid simplicial complexes are pure O -sequences. Our method consists of a new and more abstract approach, which shifts the focus from working on constructing suitable artinian level monomial ideals, as often done in the past, to the study of properties of pure O -sequences. We propose a conjecture on pure O -sequences and settle it in small socle degrees. This allows us to prove Stanley’s conjecture for all matroids of rank 3. At the end of the paper, using our method, we discuss a first possible approach to Stanley’s conjecture in full generality. Our technical work on pure O -sequences also uses very recent results of the third author and collaborators.

1. INTRODUCTION

Matroids appear in many different areas of mathematics, often in surprising or unexpected ways [18, 20, 30, 31]. Finite matroids can be naturally identified with a class of finite simplicial complexes, and are therefore also objects of great interest in combinatorial commutative algebra and algebraic combinatorics. The algebraic theory of matroids and the theory of pure O -sequences both began with Stanley’s seminal work [23]. The goal of this paper is to contribute to the study of an intriguing connection between these two fields conjectured by Stanley:

Conjecture 1.1 ([23, 24]). *The h -vector of a matroid complex is a pure O -sequence.*

This conjecture, as Proudfoot pointedly stated in [21], “...has motivated much of the [recent] work on h -vectors of matroid complexes”. Over thirty years later, Stanley’s conjecture is still wide open and mostly not understood, although a number of interesting partial results have been obtained; see [5, 6, 11, 15, 16, 19, 22, 26, 27, 28, 29]. As of today, the typical approach to Stanley’s conjecture has been to, given the h -vector h of a matroid, explicitly construct a pure monomial order ideal (or equivalently, an artinian level monomial algebra; see [2]) with h -vector h . Our goal, assisted by recent progress on pure O -sequences and especially the Interval Property in socle degree 3 (see [2]), is to avoid the above constructions and instead begin the study of Stanley’s conjecture under a new and more abstract perspective.

Our approach essentially consists of reducing ourselves to focusing, as much as possible, on properties of pure O -sequences. We formulate a new conjecture on pure O -sequences implying Stanley’s for matroids satisfying certain hypotheses, which, by induction, gives us the key to completely resolve Stanley’s conjecture for matroids of rank 3 (i.e., dimension 2). We conclude the paper by outlining, using our approach, a first, if still tentative, plan of attack to the general case of Stanley’s conjecture. Finally, we wish to point out that about five months after our submission, De Loera, Kemper and Klee [8] provided another

Key words and phrases. Pure O -sequence. Matroid complex. h -vector. Order ideal. Simplicial complex. Interval property. Monomial algebra. Complete intersection. Differentiable O -sequence.

2010 *Mathematics Subject Classification.* Primary: 05B35. Secondary: 05E45, 05E40, 13H10, 13D40.

proof of Stanley's conjecture in rank 3. Their argument for this result, although it uses the constructive approach and appears to be *ad hoc*, is simpler than ours.

2. DEFINITIONS AND PRELIMINARY RESULTS

In this section, we introduce the notation and terminology used in the paper, and state some auxiliary results. For the unexplained algebraic terminology we refer the reader to [4, 17, 24].

Simplicial and matroid complexes. Let $V = \{v_1, \dots, v_n\}$ be a set of distinct elements. A collection, Δ , of subsets of V is called a *simplicial complex* if for each $F \in \Delta$ and $G \subseteq F$, $G \in \Delta$.

Elements of the simplicial complex Δ are called *faces* of Δ . Maximal faces (under inclusion) are called *facets*. If $F \in \Delta$ then the *dimension of F* is $\dim F = |F| - 1$. The *dimension of Δ* is defined to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The complex Δ is said to be *pure* if all its facets have the same dimension.

If $\{v\} \in \Delta$, then we call v a *vertex* of Δ (we will typically ignore the distinction between $\{v\} \in \Delta$ and $v \in V$). Throughout the paper, Δ will denote a simplicial complex with vertices $\{1, \dots, n\}$.

Let $d - 1 = \dim \Delta$. The *f -vector* of Δ is the vector $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$, where $f_i = |\{F \in \Delta \mid \dim F = i\}|$ is the number of i -dimensional faces in Δ .

Let k be a field. We can associate, to a simplicial complex Δ , a squarefree monomial ideal in $S = k[x_1, \dots, x_n]$,

$$I_\Delta = \left(x_F = \prod_{i \in F} x_i \mid F \notin \Delta \right) \subseteq S.$$

The ideal I_Δ is called the *Stanley-Reisner ideal* of Δ , and the quotient algebra $k[\Delta] = S/I_\Delta$ the *Stanley-Reisner ring* of Δ .

If $W \subseteq V$ is a subset of the vertices then we define the *restriction* of Δ to W , denoted by $\Delta|_W$, to be the complex whose faces are the faces of Δ which are contained in W .

A simplicial complex Δ over the vertices V is called a *matroid complex* if for every subset $W \subseteq V$, $\Delta|_W$ is a pure simplicial complex (see, e.g., [24]). There are several equivalent definitions of a matroid complex. The one we will be using most often is the following, given by the *circuit exchange property*: Δ is a matroid complex if and only if, for any two minimal generators M and N of I_Δ , their least common multiple divided by any variable in the support of both M and N is in I_Δ .

Although we mainly use the language of algebraic combinatorics or commutative algebra in this paper, it is useful to recall here that, for most algebraic definitions or properties concerning matroids, there is also a corresponding standard formulation in matroid theory (see for instance [20, 30]). In particular, the faces of Δ are also called *independent sets*, and the non-faces are the *dependent sets*. A facet of Δ is known in matroid theory as a *basis*, and a minimal non-face of the complex (or *missing face*, as in [25]) is a *circuit*, which corresponds bijectively to a minimal monomial generator of the ideal I_Δ . The *rank* of Δ is equal to $\dim \Delta + 1$ (i.e., it is the cardinality of a basis).

Hilbert functions, h -vectors and pure O -sequences. For a standard graded k -algebra $A = \bigoplus_{n \geq 0} A_n$, the *Hilbert function* of A indicates the k -vector space dimensions of the

graded pieces of A ; i.e., $H_A(i) = \dim_k A_i$. The *Hilbert series* of A is the generating function of its Hilbert function, $\sum_{i=0}^{\infty} H_A(i)t^i$.

It is a well-known result of commutative algebra that for a homogeneous ideal $I \subseteq S$, the Hilbert series of S/I is a rational function with numerator $\sum_{i=0}^e h_i t^i$; we call the sequence $h_{S/I} = (h_0, \dots, h_e)$ the h -vector of S/I . The index e is the *socle degree* of S/I .

Define the h -vector of a simplicial complex Δ to be the h -vector of its Stanley-Reisner ring $k[\Delta]$. Of course, h -vectors are also fundamental objects in algebraic combinatorics, and are studied in a number of areas, including in the context of shellings (see e.g. [24]). Assuming that $\dim \Delta = d - 1$, the h -vector $h(\Delta) = (h_0, \dots, h_d)$ of Δ can also be derived (or, in fact, defined) from its f -vector $f(\Delta) = (f_{-1}, \dots, f_{d-1})$, via the relation

$$\sum_{i=0}^d f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^d h_i t^i.$$

In particular, for any $j = 0, \dots, d$, we have

$$(1) \quad \begin{aligned} f_{j-1} &= \sum_{i=0}^j \binom{d-i}{j-i} h_i; \\ h_j &= \sum_{i=0}^j (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}. \end{aligned}$$

Unlike the algebraic case, we allow the h -vector of a simplicial complex to end with some trailing 0's.

A standard graded algebra A is *artinian* if its Hilbert function H_A is eventually zero (see, e.g., [4, 24] for a number of equivalent definitions). Notice that, in the artinian case, the Hilbert function, H_A , of A can be naturally identified with its h -vector, h_A . A sequence that occurs as the h -vector of some artinian standard graded algebra is called an O -sequence (see [4, 14] for Macaulay's characterization of the O -sequences). We will not need to go further into this here, but it is helpful to point out that, from an algebraic point of view, the context of artinian algebras is exactly the one we will implicitly be working in in the main portion of the paper.

Let us also recall that the h -vector h of a matroid Δ can also be expressed in terms of the *Tutte polynomial*, $T(x, y)$, of Δ ; precisely, we have $T(x, 1) = h_0 x^d + h_1 x^{d-1} + \dots + h_d$ (see for instance [1]). Furthermore, the h -vector h of any matroid is an O -sequence, as can be shown using standard tools from commutative algebra. (In fact, all matroid h -vectors are level h -vectors; see e.g. [24].) Notice, in particular, that h is nonnegative, a fact not obvious *a priori* and that was first proved combinatorially.

An algebra S/I is a *complete intersection* if its codimension is equal to the number of minimal generators of I . In the case of matroid (or, more generally, arbitrary monomial) ideals I_Δ , it is easy to see that I_Δ is a complete intersection if and only if the supports of its minimal generators are pairwise disjoint. Complete intersection matroids are those that in matroid theory are the *connected sums* of their circuits. Equivalently, they are the join of boundaries of simplices, or in the language of commutative algebra, they coincide with *Gorenstein* matroids (see [26]).

A finite, nonempty set X of (monic) monomials in the indeterminates y_1, y_2, \dots, y_r is called a (*monomial*) *order ideal* if whenever $M \in X$ and N is a monomial dividing M , $N \in X$. Notice that X is a ranked poset with respect to the order given by divisibility. The

h-vector $h = (h_0 = 1, h_1, \dots, h_e)$ of X is its rank vector; in other words, h_i is the number of monomials of X having degree i . We say that X is *pure* if all of its maximal monomials have the same degree. The sequences that occur as *h*-vectors of order ideals are, in fact, precisely the *O*-sequences defined above. A *pure O-sequence* is the *h*-vector of a pure order ideal.

Deletions, links and cones. Let $v \in V$. The *deletion* of v from a simplicial complex Δ , denoted by Δ_{-v} , is defined to be the restriction $\Delta|_{V-\{v\}}$. The *link* of v in Δ , denoted by $\text{link}_\Delta(v)$, is the simplicial complex $\{G \in \Delta \mid v \notin G, G \cup \{v\} \in \Delta\}$. For simplicity in dealing with Stanley-Reisner ideals, we will also consider links and deletions to be complexes defined over V . In particular, not all elements of the vertex set will be faces of a link or a deletion complex. Link and deletion are identical to the *contraction* and *deletion* constructions from matroid theory.

Let $x \notin V$ be a new vertex. The *cone* over the simplicial complex Δ with *apex* x is the simplicial complex $\{F \cup \{x\} \mid F \in \Delta\} \cup \{F \in \Delta\}$. Notice that a complex Γ is a cone with apex x if and only if x is contained in all the facets of Γ . A matroid is a cone if and only if it has a *coloop*, which corresponds to the apex defined above.

The following useful facts are standard (see for instance [26]).

Remark 2.1. Let Δ be a matroid complex of dimension $d - 1$, and let $v \in \Delta$. Then:

- (1) Δ_{-v} is a matroid complex, and its Stanley-Reisner ideal is $I_{\Delta_{-v}} = I_\Delta + (x_v)$.
- (2) $\text{link}_\Delta(v)$ is a matroid complex, and its Stanley-Reisner ideal is $I_{\text{link}_\Delta(v)} = (I_\Delta : x_v) + (x_v)$.
- (3) (The nonzero portion of) $h(\Delta)$ coincides with (the nonzero portion of) the *h*-vector of any cone over Δ .
- (4) $h(\Delta)$ has socle degree d (i.e., $h_d \neq 0$) if and only if Δ is not a cone.

Finally, we recall the well-known Brown-Colbourn inequalities on matroid *h*-vectors:

Lemma 2.2 ([3]). *Let $h = (h_0, h_1, \dots, h_d)$ be a matroid *h*-vector. Then, for any index j , $0 \leq j \leq d$, and for any real number $\alpha \geq 1$, we have*

$$(-1)^j \sum_{i=0}^j (-\alpha)^i h_i \geq 0$$

(where the inequality is strict for $\alpha \neq 1$).

3. THE CONJECTURE ON PURE *O*-SEQUENCES

In this section, we propose a conjecture on pure *O*-sequences and prove this conjecture for small socle degrees. This result will be crucial to settle Stanley's conjecture for matroids of rank 3 in the next section. Our argument is by induction on the link and deletion of Δ . Since they have fewer vertices, $h(\Delta_{-v})$ and $h(\text{link}_\Delta(v))$ are both pure *O*-sequences by induction. The *h*-vector of Δ can be computed as the shifted sum of $h(\Delta_{-v})$ and $h(\text{link}_\Delta(v))$, namely,

$$h_i(\Delta) = h_i(\Delta_{-v}) + h_{i-1}(\text{link}_\Delta(v)),$$

for all i . We conjecture conditions (which are satisfied by rank 3 matroids) that imply that the shifted sum of two pure *O*-sequences is a pure *O*-sequence.

Given a vector $H = (1, H_1, H_2, \dots, H_t)$ of natural numbers, define $\Delta H = (1, H_1 - 1, H_2 - H_1, \dots, H_t - H_{t-1})$ to be its *first difference*. H is *differentiable* if ΔH is an *O*-sequence. (It is easy to see that a differentiable vector is itself an *O*-sequence.)

Our conjecture is stated as follows.

Conjecture 3.1. *Let $h = (1, h_1, \dots, h_e)$ and $h' = (1, h'_1, \dots, h'_{e-1})$ be two pure O -sequences, and suppose that $(\Delta h')_i \leq (\Delta h)_i$ for all $i \leq \lceil e/2 \rceil$, and $h'_i \leq h_i$ for all $i \leq e - 1$. Then the shifted sum of h and h' ,*

$$h'' = (1, h_1 + 1, h_2 + h'_1, \dots, h_e + h'_{e-1}),$$

is also a pure O -sequence.

Remark 3.2. It is proved in [2, Theorem 3.3] that all socle degree 3 nondecreasing pure O -sequences are differentiable. (This fact is false for higher socle degrees; see [2, Proposition 3.5].) This justifies the assumption, in Conjecture 3.1, of the inequality between the initial parts of the first differences of h and h' . For example, $h = (1, 6, 6, 6)$ and $h' = (1, 3, 6)$ are two pure O -sequences satisfying the condition $h'_i \leq h_i$ for all $i \leq e - 1$, but $(\Delta h')_2 = 3 \not\leq (\Delta h)_2 = 0$; and in this case, their shifted sum, $h'' = (1, 7, 9, 12)$, is not differentiable, hence not a pure O -sequence.

We first need some lemmas. The first is a theorem of Hibi and Hausel on pure O -sequences, a result which is analogous to Swartz's (algebraic) g -theorem for matroids [28].

Lemma 3.3 (Hausel, Hibi). *Let $h = (1, h_1, \dots, h_e)$ be a pure O -sequence. Then h is flawless (that is, $h_i \leq h_{e-i}$ for all $i \leq e/2$), and its “first half”, $(1, h_1, \dots, h_{\lceil e/2 \rceil})$, is a differentiable O -sequence.*

Proof. See [10, Theorem 6.3]. Differentiability is due to Hausel, who, in fact, proved an (algebraic) g -theorem for pure O -sequences in characteristic zero. That any pure O -sequence is flawless, and (consequently) nondecreasing throughout its first half, was first shown by Hibi [12]. (The part of the result due to Hibi will actually be enough for our purposes here.) \square

The following conjecture is referred to as the Interval Conjecture for Pure O -sequences (ICP), and was recently stated by the last author in collaboration with Boij, Migliore, Mirò-Roig and Nagel [2] (see [32] for the original formulation of the Interval Conjectures in the context of arbitrary level and Gorenstein algebras, where it is still wide open). In [2], the ICP was proved for socle degrees at most 3, which will be a crucial tool in our proof. We should also point out that, however, while the ICP remains open in most instances — for example, in three variables — it has recently been disproved in the four variable case by A. Constantinescu and M. Varbaro [7].

Conjecture 3.4 ([2, Conjecture 4.1]). *Suppose that, for some positive integer α , both $(1, h_1, \dots, h_i, \dots, h_e)$ and $(1, h_1, \dots, h_i + \alpha, \dots, h_e)$ are pure O -sequences. Then $(1, h_1, \dots, h_i + \beta, \dots, h_e)$ is also a pure O -sequence for each integer $\beta = 1, 2, \dots, \alpha - 1$.*

Lemma 3.5 ([2, Theorem 4.3]). *The ICP holds for $e \leq 3$.*

The next proposition on differentiable O -sequences is essential in proving Conjecture 3.1 for socle degrees at most 3. While it is possible to show it using Macaulay's classification of O -sequences (see [4]), the required arguments are lengthy. We thank an anonymous reader for suggesting the shorter proof given below. We start with a lemma, and then record the proposition only in socle degree 3, which is the case we are interested in here.

Lemma 3.6. *Let h and h' be O -sequences such that $h'_i \leq h_i$ for all $i \leq e - 1$. Then $h''_i = h_i + h'_{i-1}$ is an O -sequence.*

Proof. Let X be the order ideal formed by taking, for each i , the last h_i monomials in the lexicographic order on some set of variables y_1, y_2, \dots, y_{h_1} . Since h is an O -sequence, X is an order ideal by Macaulay's theorem ([4]) and has h -vector h . Construct similarly another order ideal, X' , to have h -vector h' . Since $h'_i \leq h_i$, we have $X' \subseteq X$.

Let t be a new variable and consider $X'' = X \cup tX'$, which is clearly an order ideal. Since $X' \subseteq X$, it is easy to see that we can calculate the h -vector of X'' as $h_i(X'') = h_i(X) + h_{i-1}(X')$, showing that h'' is an O -sequence. \square

Proposition 3.7. *Let $h = (1, r-1, a, b)$ and $h' = (1, r', c)$ be two differentiable O -sequences such that $(\Delta h')_i \leq (\Delta h)_i$ for $i \leq 2$. Then the shifted sum of h and h' , $h'' = (1, r, a+r', b+c)$, is also a differentiable O -sequence.*

Proof. Since h and h' are differentiable, Δh and $\Delta h'$ are O -sequences. The hypotheses allow us to apply Lemma 3.6 to the first differences. Thus the shifted sum of Δh and $\Delta h'$ is an O -sequence. Clearly the shifted sum of Δh and $\Delta h'$ is equal to the first difference of the shifted sum of h and h' , which is h'' . Since $\Delta h''$ is an O -sequence, h'' is a differentiable O -sequence. \square

Lemma 3.8. *Let t and r be positive integers. Then $h = (1, r, r, t)$ is a pure O -sequence if and only if $\lceil r/3 \rceil \leq t \leq r$.*

Proof. See the proof of [2, Theorem 5.8]. \square

We are now ready to prove our main result of this section.

Theorem 3.9. *Conjecture 3.1 holds in socle degrees $e \leq 3$.*

Proof. The case $e = 1$ is trivial. Suppose that $e = 2$. We want to show that, if $(1, r-1, a)$ and $(1, r')$ are pure O -sequences and $1 \leq r' \leq r-1$, then $h = (1, r, a+r')$ is also pure. But by [2, Corollary 4.7], $(1, r-1, a)$ being pure means that $\lceil (r-1)/2 \rceil \leq a \leq \binom{r}{2}$. We have

$$\lceil (r+1)/2 \rceil = \lceil (r-1)/2 \rceil + 1 \leq a + r' \leq \binom{r}{2} + (r-1) = \binom{r+1}{2} - 1,$$

and the result follows by invoking again [2, Corollary 4.7].

We now turn to the case $e = 3$. Let $h = (1, r-1, a, b)$ and $h' = (1, r', c)$ be pure O -sequences satisfying the hypotheses of Conjecture 3.1. Most importantly, since $\lceil 3/2 \rceil = 2$, $(\Delta h')_i \leq (\Delta h)_i$, for $i \leq 2$. Our goal is to show that $h'' = (1, r, a+r', b+c)$ is also a pure O -sequence.

Notice that, by Lemma 3.3, $a \geq r-1$, and therefore $a+r' \geq r$. Also, given a and r , by [2, Theorem 3.3 and Corollary 3.2], the maximum value, B , that b may assume in the pure O -sequence $h = (1, r-1, a, b)$ coincides with the maximum b that makes h differentiable. The same is clearly true for the maximum value, C , of c in h' . We consider two cases depending on whether $b+c \geq a+r'$.

If $b+c \geq a+r'$ then all values of $b+c$ making h'' a pure O -sequence are in the range $[a+r', B+C]$. Furthermore, $h = (1, r-1, a, B)$ and $h' = (1, r', C)$ being differentiable implies, by Proposition 3.7, that their shifted sum, $(1, r, a+r', B+C)$, is also a differentiable O -sequence. From Macaulay's theorem [4, 14] it easily follows that h'' is differentiable for all values of $b+c \in [a+r', B+C]$. By [2, Corollary 3.2], every finite differentiable O -sequence is pure and thus h'' is pure.

Assume now that $b+c < a+r'$. Let W be a pure order ideal of monomials in the variables y_1, \dots, y_{r-1} generated by M_1, \dots, M_b and having h -vector h , and let W' be generated by

monomials N_1, \dots, N_c , in the variables $y_1, \dots, y_{r'}$, and have h -vector h' . Consider the pure order ideal W'' generated by the $b+c$, degree 3 monomials $M_1, \dots, M_b, y_r N_1, \dots, y_r N_c$ (which contain all the variables y_1, \dots, y_r). Since each of the $y_1, \dots, y_{r'}$ appears in some N_i , all the r' monomials $y_r y_1, \dots, y_r y_{r'}$ must appear among the degree 2 monomials of W'' , along with (at least) the a divisors of the M_i 's. It follows that the h -vector of W'' is $h_{W''} = (1, r, a_1, b+c)$, for some $a_1 \geq a + r'$. In particular, $h_{W''}$ is a pure O -sequence.

Consider the case when $b+c \leq r$. Since any degree 3 monomial has at most three degree 2 monomial divisors, we have $b+c \geq \lceil (a+r')/3 \rceil \geq \lceil r/3 \rceil$. Therefore, by Lemma 3.8, $(1, r, r, b+c)$ is a pure O -sequence. Because $r \leq a+r' \leq a_1$, it follows by employing again Lemma 3.5 that h'' is pure, as desired.

It remains to consider the case when $r < b+c < a+r'$. Let a_0 be the least integer (depending on r and $b+c$) such that $h_A = (1, r, a_0, b+c)$ is a differentiable O -sequence. It is easy to see that, under the current assumptions, a_0 always exists and satisfies $a_0 \leq b+c < a+r' \leq a_1$. Since by [2, Corollary 3.2] h_A is pure, Lemma 3.5 (applied to the interval defined in degree 2 by h_A and $h_{W''}$) gives that $h'' = (1, r, a+r', b+c)$ is a pure O -sequence. This concludes the proof of the theorem. \square

4. STANLEY'S CONJECTURE FOR RANK 3

The goal of this section is to settle Stanley's conjecture for matroids of rank 3 (or dimension 2).

Remark 4.1. It is a well-known fact (see, e.g., [4]) that the h -vector of a complete intersection S/I is entirely determined by the degrees of the generators of I . In fact, given the h -vector h of a complete intersection S/I , where I is generated in degrees d_1, \dots, d_t , it is a standard exercise to show (for instance using *Macaulay's inverse systems*; see [9, 13] for an introduction to this theory) that h is the pure O -sequence given by the order ideal whose unique maximal monomial is $y_1^{d_1-1} \dots y_t^{d_t-1}$. In particular, it follows that Stanley's conjecture holds for the class of all complete intersection matroids.

The next few lemmas will be technically essential to prove our main result. Lemma 4.2, (1) states a well-known fact in matroid theory; we include a brief argument for completeness. Lemma 4.2, (2) was erroneously stated in [26] in a remark without the assumption that $\dim \Delta \leq 2$. We say that two vertices $i, j \in \Delta$ are in *series* if for every minimal generator $u \in I_\Delta$, $x_i | u$ if and only if $x_j | u$. A maximal set of vertices with each pair in series is called a *series class*.

Lemma 4.2. *Let Δ be a matroid complex that is not a cone. Then:*

- (1) *For any $v \in \Delta$, $\text{link}_\Delta(v)$ is not a cone.*
- (2) *Assume that $d = \dim \Delta \leq 2$ and that, for each vertex $w \in \Delta$, Δ_{-w} is a cone. Then Δ is a complete intersection.*

Proof. (1) Suppose that $\Gamma = \text{link}_\Delta(v)$ is a cone, say with apex w . Then, by the purity of Δ and Γ , F is a facet of Γ if and only if $F \cup \{v\}$ is a facet of Δ . It follows that for any facet G of Δ containing v , we have $w \in G$ and $G - \{w\} \in \Delta_{-w}$. Since Δ is a matroid and not a cone, Δ_{-w} is a pure complex of the same dimension as Δ . Thus $G - \{w\}$ is contained in a facet of Δ_{-w} , say H , of dimension equal to the dimension of G (which is also the dimension of Δ). Therefore H is also a facet of Δ . But since H contains v , it must also contain w , which is a contradiction.

(2) Since Δ_{-w} is a cone for any vertex w , it is a standard fact that the vertices of Δ can be partitioned into series classes, say S_1, \dots, S_k , where (since Δ is not a cone) $|S_i| \geq 2$ (see e.g. [6, 26]). Also, each facet in Δ contains at least $|S_i| - 1 \geq 1$ elements in S_i , for each i . If $\text{rank } \Delta = 2$, then $k = 1$ or $k = 2$. If $k = 1$ then $|S_1| = 3$ and Δ is the boundary of a 2-simplex. If $k = 2$ then $|S_1| = |S_2| = 2$ and Δ is a 4-cycle. Both of these are complete intersections.

If $\text{rank } \Delta = 3$, we have $k \leq 3$. If $k = 1$, then $|S_1| = 4$ (and Δ is the boundary complex of a tetrahedron). If $k = 2$, since each facet of Δ has 3 elements, the only possibility (after re-indexing) is that $|S_1| = 3$ and $|S_2| = 2$ (that is, Δ is a bi-pyramid over an unfilled triangle). If $k = 3$, then similarly we must have $|S_1| = |S_2| = |S_3| = 2$ (and Δ is the boundary complex of an octahedron). In any case, it can easily be seen that I_Δ is a complete intersection, as desired. \square

Example 4.3. The assumption $\dim \Delta \leq 2$ is necessary in Lemma 4.2. The smallest example of a dimension 3 matroid Δ which is not a cone or a complete intersection, and such that the deletion of any vertex of Δ yields a cone, has Stanley-Reisner ideal

$$I_\Delta = (x_1x_2x_5x_6, x_1x_2x_3x_4, x_3x_4x_5x_6) \subseteq k[x_1, x_2, x_3, x_4, x_5, x_6].$$

Notice that the h -vector of Δ is $h'' = (1, 2, 3, 4, 2)$, which can easily be seen to satisfy Stanley's conjecture.

Notation. In what follows, for a simplicial complex Ω , $\text{init } I_\Omega$ indicates the smallest degree of a minimal non-linear generator of I_Ω . For matroid complexes this is also the smallest cardinality of a circuit that is not a loop.

Lemma 4.4. *Suppose Δ is a 2-dimensional matroid complex with $\text{init } I_\Delta \geq 3$. Then Δ satisfies Stanley's conjecture.*

Proof. The h -vector of Δ is of the form $h = (1, r, \binom{r+1}{2}, h_3)$. Since Δ is matroid, we may assume that $h_3 > 0$ (i.e., Δ is not a cone), otherwise the result is trivial. From the Brown-Colbourn inequalities (Lemma 2.2) with $j = d = 3$ and $\alpha = 1$, we obtain

$$h_3 \geq \binom{r+1}{2} - r + 1 = \binom{r}{2} + 1.$$

Since, clearly, $h_3 \leq \binom{r+2}{3}$ and $(1, r, \binom{r+1}{2}, \binom{r+2}{3})$ is a pure O -sequence, by Lemma 3.5 it suffices to show that $H = (1, r, \binom{r+1}{2}, \binom{r}{2} + 1)$ is a pure O -sequence. Let us consider the pure order ideal X in variables y_1, \dots, y_r whose maximal monomials are y_r^3 and the $\binom{r}{2}$ monomials of the form $y_r \cdot M$, where M ranges among all degree 2 monomials in y_1, \dots, y_{r-1} . One moment's thought gives that the h -vector of X is indeed H , as desired. \square

Remark 4.5. It can be proved with a considerably more technical argument that, in fact, under the hypotheses of the previous lemma, $h_3 = 0$ or $h_3 \geq \binom{r+1}{2} - 1$.

Given a simplicial complex Ω , we use $[\Omega]_i$ to denote its i -skeleton, that is, the simplicial complex given by the faces of Ω of dimension at most i .

Lemma 4.6. *Let Δ be a 2-dimensional matroid complex with $\text{init } I_\Delta = 2$. Then, for any vertex v such that x_v divides a minimal generator of I_Δ of degree 2, we have, for $i \leq 2$,*

$$h(\text{link}_\Delta(v))_i \leq h(\Delta_{-v})_i$$

and

$$\Delta h(\text{link}_\Delta(v))_i \leq \Delta h(\Delta_{-v})_i.$$

Proof. Clearly, the desired inequalities on the h -vectors follow from those on their first differences, so we only need to show that, for $i \leq 2$,

$$(2) \quad \Delta h(\text{link}_\Delta(v))_i \leq \Delta h(\Delta_{-v})_i.$$

Since $\dim \Delta = 2$, then the codimension of $k[\Delta]$ is $r = n - 3$, where as usual n is the number of vertices in Δ . Consider first the case where $I = I_\Delta$ has no degree 3 generators. Let $J = I_{\langle 2 \rangle}$ be the ideal generated by the degree 2 generators of I , and let Γ be its corresponding complex. Then Γ is a matroid. Indeed, by the circuit exchange property, we only need to notice that if $x_i x_j, x_j x_k \in J$, then $x_i x_k$ is a minimal generator of I , and thus is in J .

Let v be any vertex such that x_v divides a minimal generator of I of degree 2. We will prove the inequalities (2) on the first three entries of the h -vectors of the link and the deletion in Γ . Assuming those, we now show that the inequalities (2) for Δ will follow. Indeed, if $\Delta = \Gamma$ (equivalently, I_Δ has no degree 4 minimal generators), then we are done. Hence suppose $\Delta \neq \Gamma$. It can easily be seen that, for any complex Ω , the first three entries of the h -vector of any of its skeletons $[\Omega]_j$ are the consecutive sums of the corresponding entries of $h([\Omega]_{j+1})$. Moreover, we have the equalities $[\text{link}_\Omega(v)]_{j-1} = \text{link}_{[\Omega]_j}(v)$ and $[\Omega_{-v}]_j = ([\Omega]_j)_{-v}$. Thus, by induction, starting from Γ , the inequalities (2) follow for any j -skeleton of Γ , and in particular for the 2-skeleton Δ , and we are also done.

Therefore, for the case where I has no degree 3 generators, it remains to show that, for $i \leq 2$,

$$(3) \quad \Delta h(\text{link}_\Gamma(v))_i \leq \Delta h(\Gamma_{-v})_i.$$

Recall that Γ is a matroid whose Stanley-Reisner ideal J is generated in degree 2. Using the circuit exchange property, there exist pairwise disjoint subsets of variables, W_1, \dots, W_t , such that J is generated by all the squarefree degree 2 monomials coming from the W_j 's (in the language of matroid theory, these latter are known as the *parallel classes* of Δ).

Let $w_j = |W_j|$. Note that $w_j \geq 2$. Without loss of generality, assume that $v \in W_1$. Then $I_{\Gamma_{-v}}$ is generated by all squarefree degree 2 monomials of $W_1 - \{v\}, W_2, \dots, W_t$, whereas $I_{\text{link}_\Gamma(v)}$ is generated by all squarefree degree 2 monomials of W_2, \dots, W_t . For small degrees, we can compute the h -vectors of Δ_{-v} and $\text{link}_\Delta(v)$ by subtracting the number of generators from the h -vectors of the corresponding polynomial rings. In particular, by letting s be the codimension of Γ , we have

$$\begin{aligned} h_1(\text{link}_\Gamma(v)) &= s - w_1 + 1, \quad h_2(\text{link}_\Gamma(v)) = \binom{s - w_1 + 2}{2} - \binom{w_2}{2} - \dots - \binom{w_t}{2}, \\ h_1(\Gamma_{-v}) &= s - 1, \quad \text{and } h_2(\Gamma_{-v}) = \binom{s}{2} - \binom{w_1 - 1}{2} - \binom{w_2}{2} - \dots - \binom{w_t}{2}. \end{aligned}$$

For $i = 1$, the inequality (3) is trivial, since $w_1 \geq 2$. For $i = 2$, it is equivalent to $\binom{s}{2} - \binom{w_1 - 1}{2} + 1 \geq \binom{s - w_1 + 2}{2} + w_1 - 1$, i.e., $\binom{s}{2} - \binom{w_1}{2} \geq \binom{s - w_1 + 2}{2} - 1$. But this is clearly the same as

$$w_1 + (w_1 + 1) + \dots + (s - 1) \geq 2 + 3 + \dots + (s - w_1 + 1),$$

which is true since $w_1 \geq 2$.

Let us now turn to the general case, where I may have degree 3 generators. Let L be the ideal obtained by adding all squarefree degree 4 monomials to $I_{\langle 2 \rangle}$ and let Δ' be the

simplicial complex associated to L . Clearly, $\dim \Delta' = 2$. Since $L_{\langle 2 \rangle} = I_{\langle 2 \rangle}$, by the circuit exchange property in I , we again have that the simplicial complex Γ' associated to $L_{\langle 2 \rangle}$ is a matroid. In particular, since Δ' is the 2-skeleton of Γ' , it is also a matroid. Notice also that since x_v divides a degree 2 generator of I , neither Δ nor Δ' is a cone with apex v . This implies that $\dim \Delta'_{-v} = \dim \Delta_{-v} = 2$. Also, by the pureness of the complexes, $\dim \text{link}_{\Delta}(v) = \dim \text{link}_{\Delta'}(v) = 1$.

Observe that any degree 3 generator u of I that does not have x_v in its support is also a degree 3 generator of $I + x_v$ and $I : x_v$; on the other hand, if the support of u contains x_v then then $\frac{u}{x_v}$ is a degree 2 element of $I : x_v$. This implies that the existence of a degree 3 generator in I does not change the first three entries of the Hilbert function of Δ_{-v} , does not change the first two entries of the Hilbert function of $\text{link}_{\Delta}(v)$, and does not decrease the third entry of the Hilbert function of $\text{link}_{\Delta}(v)$.

It now follows that $h_i(\Delta'_{-v}) = h_i(\Delta_{-v})$ for $i \leq 2$, $h_1(\text{link}_{\Delta'}(v)) = h_1(\text{link}_{\Delta}(v))$ and $h_2(\text{link}_{\Delta'}(v)) \geq h_2(\text{link}_{\Delta}(v))$. Since $h_i(\Delta) = h_i(\Delta_{-v}) + h_{i-1}(\text{link}_{\Delta}(v))$, we have $h_1(\Delta) = h_1(\Delta')$, and it suffices to prove the inequalities (2) for Δ' . But this is true by the previous case, where the matroid had no degree 3 generators. This concludes the proof of the lemma. \square

We are now ready to establish Stanley's conjecture for matroids of rank 3.

Theorem 4.7. *Stanley's conjecture holds for rank 3 (i.e., 2-dimensional) matroids.*

Proof. We may assume that Δ is not a complete intersection, since Corollary 4.1 already took care of this case. We may also assume that Δ is not a cone since Stanley's conjecture holds for 1-dimensional matroids (see, e.g., [2, 26, 27]). Moreover, the case $\text{init } I_{\Delta} \geq 3$ has been dealt with in Lemma 4.4. Thus, we will assume that $\text{init } I_{\Delta} = 2$.

By Lemma 4.2, there exists a vertex v such that Δ_{-v} and $\text{link}_{\Delta}(v)$ are not cones. Additionally, we claim that with essentially one exception, we can also choose a vertex v so that x_v divides a degree 2 minimal generator of I_{Δ} .

Suppose that we cannot choose such a vertex v . Then for every minimal generator $x_i x_j \in I_{\Delta}$, Δ_{-i} and Δ_{-j} must both be cones (otherwise, replace v by i or j). That is, $S = \{i, j\}$ is a pair of *parallel elements* each belonging to a nontrivial (i.e., of cardinality at least 2) series class. It is a standard fact in matroid theory that this is the case only if these series classes contain S , and Δ is the join (i.e., the *direct sum*) of the restrictions of Δ to S and to its complement, \bar{S} , in Δ .

Suppose $I_{\Delta|_{\bar{S}}} = I_{\bar{S}}$ has a minimal generator of degree 2, say $x_k x_l$. Hence, by a similar argument, $\{k, l\}$ belongs to a series class, and $\Delta|_{\bar{S}}$ is the join of $\{\{k\}, \{l\}\}$ with a dimension 0 matroid. If the complement of $\{k, l\}$ in \bar{S} contains exactly 1 vertex, then Δ is a complete intersection. If this complement has at least 2 vertices, then since each facet of Δ must contain at least 1 vertex from each series class, $\Delta|_{\bar{S}}$ must be a square. Thus, Δ is the boundary complex of an octahedron, which is again a complete intersection. Now assume that $I_{\bar{S}}$ has no minimal generators of degree 2. Since $\Delta|_{\bar{S}}$ is a dimension 1 matroid, this implies that, after a re-indexing, $I_{\Delta} = (x_1 x_2) + J$, where J is the ideal generated by all squarefree degree 3 monomials in $\{x_3, \dots, x_n\}$, and $n \geq 6$ (if $n = 5$ then $(x_1 x_2) + J$ is a complete intersection).

Let Γ be the simplicial complex corresponding to J , and let $I = I_{\Delta}$. Observe that $I + (x_1) = J + (x_1)$ and $I : x_1 = J + (x_2)$ are isomorphic and have the same h -vector as J . Thus, the h -vector of Δ is the shifted sum of $h(\Gamma)$ with itself. It is easy to compute that

$h(\Gamma) = (1, r-1, \binom{r}{2})$, and so

$$h(\Delta) = \left(1, r, \binom{r}{2} + r - 1, \binom{r}{2}\right).$$

In order to prove that $h(\Delta)$ is a pure O -sequence, consider the pure order ideal in variables y_1, \dots, y_r whose maximal monomials have the form $y_i \cdot M$, where M is a degree 2 squarefree monomial and i is the smallest index of a variable dividing M . These are $\binom{r}{2}$ monomials of degree 3, and dividing by a variable, they give all degree 2 monomials in y_1, \dots, y_r except y_r^2 . The number of these degree 2 monomials is $\binom{r+1}{2} - 1 = \binom{r}{2} + r - 1$. This proves that $h(\Delta)$ is a pure O -sequence. As a consequence, this shows that Stanley's conjecture holds in the exceptional case where a vertex v as claimed could not be picked.

Now assume that there exists a vertex v such that Δ_{-v} and $\text{link}_\Delta(v)$ are not cones, and x_v divides a minimal degree 2 generator of I_Δ . Notice that $I_{\Delta_{-v}}$ has fewer generators of degree 2 than I_Δ . By induction on the number of degree 2 generators, $h(\Delta_{-v})$ is a pure O -sequence. Since $\text{link}_\Delta(v)$ is a matroid of dimension 1, $h(\text{link}_\Delta(v))$ is also a pure O -sequence. Moreover, by Lemma 4.6, the h -vectors of $\text{link}_\Delta(v)$ and Δ_{-v} satisfy the hypotheses of Theorem 3.9. Therefore $h(\Delta)$ is a pure O -sequence, which concludes the proof of the theorem. \square

Our approach to prove Stanley's conjecture in dimension 2 consisted of showing that, for all matroids outside some special classes for which we could control the h -vectors, the h -vectors of link and deletion with respect to a suitably chosen vertex satisfy the hypotheses of Conjecture 3.1, which in turn we proved in socle degree 3. In a similar fashion, assuming Conjecture 3.1 is true in general, it can be seen that Stanley's conjecture holds for all matroid complexes Δ in a set \aleph defined inductively by the following two conditions:

- (i) Δ is not a cone.
- (ii) If Δ is not a complete intersection, there exists a vertex v of Δ such that $\text{link}_\Delta(v)$ and Δ_{-v} are both in \aleph , and the h -vectors h' of $\text{link}_\Delta(v)$ and h of Δ_{-v} satisfy the hypotheses of Conjecture 3.1.

We conclude our paper by briefly outlining a possible future research direction to finally tackle Stanley's conjecture in full generality, using the method of this paper. In order to generalize our result from rank 3 to the arbitrary case, one now wants to find a reliable assumption on pure O -sequences which implies Stanley's conjecture after being applied inductively on all matroid h -vectors (with the possible exception of some special class of matroids for which it is possible to control the h -vectors).

Let us assume that Stanley's conjecture holds for all matroids whose deletions with respect to any vertex are cones (for instance, this is not too difficult to show in rank 4 with arguments very similar to those of this paper). Then, using our approach, we easily have that Stanley's conjecture holds in general if, for example, the following two natural (but still too bold?) assumptions are true:

- (a) *A matroid h -vector is differentiable for as long as it is nondecreasing.*
- (b) *Suppose the shifted sum h'' of two pure O -sequences is differentiable for as long as it is nondecreasing. Then h'' is a pure O -sequence.*

Notice that part (b), if true, appears to be difficult to prove in arbitrary socle degree, given that very little is known on the "second half" of a pure O -sequence, and that unlike the first half, this can behave very pathologically (see [2]). As for part (a), it would also be of considerable independent interest to show this fact *algebraically*. That is, proving that a

g -element, that Swartz showed to exist up until the first half of a matroid h -vector (see [28] for details), does in fact carry on for as long as the h -vector is increasing.

ACKNOWLEDGMENTS

We warmly thank Ed Swartz for a number of extremely useful comments on a previous version of this work, which in particular helped us make some of the technical results on matroids simpler and better-looking. We also wish to thank an anonymous reader who suggested a much shortened proof of Proposition 3.7, and the referee for very helpful comments. Part of this work was developed in Spring 2009 and Spring 2010 during visits of the first and the third authors to Michigan Tech and Tulane, respectively. The two authors wish to thank those institutions for their support and hospitality. The first author is partially supported by the Board of Regents grant LEQSF(2007-10)-RD-A-30.

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